

PELL'S EQUATION AND SERIES EXPANSIONS FOR IRRATIONAL NUMBERS

^{1,2}CHUANAN WEI

¹*Department of Mathematics
Shanghai Normal University, Shanghai 200234, China*

²*Department of Information Technology
Hainan Medical College, Haikou 571199, China*

ABSTRACT. Solutions of Pell's equation and hypergeometric series identities are used to study series expansions for \sqrt{p} where p are arbitrary prime numbers. Numerous fast convergent series expansions for this family of irrational numbers are established.

1. INTRODUCTION

Pell's equation(also called the Pell-Fermat equation) is any Diophantine equation of the form

$$x^2 - py^2 = 1, \quad (1)$$

where p is a given positive nonsquare integer and integer solutions are sought for x and y . Joseph Louis Lagrange proved that Pell's equation has infinitely many distinct integer solutions. Furthermore, there holds the following relation.

Lemma 1. *Let s be a positive integer. If (x_1, y_1) is the integer solution to (1), then (x_s, y_s) with*

$$\begin{cases} x_s = \sum_{k=0}^{\lfloor \frac{s}{2} \rfloor} \binom{s}{2k} p^k y_1^{2k} x_1^{s-2k}, \\ y_s = \sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} \binom{s}{1+2k} p^k y_1^{1+2k} x_1^{s-1-2k} \end{cases}$$

is also the integer solution of (1).

Proof. Because that (x_1, y_1) is the integer solution to (1), we obtain

$$(x_1 + \sqrt{p} y_1)^s (x_1 - \sqrt{p} y_1)^s = 1.$$

In terms of the binomial theorem

$$(u + v)^t = \sum_{k=0}^t \binom{t}{k} u^k v^{t-k},$$

2010 Mathematics Subject Classification: Primary 65B10 and Secondary 33C20.

Key words and phrases. Pell's equation; Hypergeometric series; Series expansions for irrational numbers.

Email address: weichuanan78@163.com.

we get the following two expansions

$$\begin{aligned}
(x_1 + \sqrt{p}y_1)^s &= \sum_{k=0}^s \binom{s}{k} (\sqrt{p}y_1)^k x_1^{s-k} \\
&= \sum_{k=0}^{\lfloor \frac{s}{2} \rfloor} \binom{s}{2k} p^k y_1^{2k} x_1^{s-2k} \\
&\quad + \sqrt{p} \sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} \binom{s}{1+2k} p^k y_1^{1+2k} x_1^{s-1-2k}, \\
(x_1 - \sqrt{p}y_1)^s &= \sum_{k=0}^s \binom{s}{k} (-\sqrt{p}y_1)^k x_1^{s-k} \\
&= \sum_{k=0}^{\lfloor \frac{s}{2} \rfloor} \binom{s}{2k} p^k y_1^{2k} x_1^{s-2k} \\
&\quad - \sqrt{p} \sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} \binom{s}{1+2k} p^k y_1^{1+2k} x_1^{s-1-2k}.
\end{aligned}$$

So we gain

$$\left\{ \sum_{k=0}^{\lfloor \frac{s}{2} \rfloor} \binom{s}{2k} p^k y_1^{2k} x_1^{s-2k} \right\}^2 - p \left\{ \sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} \binom{s}{1+2k} p^k y_1^{1+2k} x_1^{s-1-2k} \right\}^2 = 1.$$

This completes the proof of Lemma 1. \square

The circumference ratio $\pi = 3.1415926535 \dots$ is one of the most important irrational numbers. For centuries, the study of π -formulas attracted many persons. Recently, Chu [4], Liu [12, 13] and Wei and Gong [14] gave many π -formulas in terms of the hypergeometric method. Different methods and results can be seen in the papers [5, 7, 8, 10, 11, 15]. For historical notes and introductory informations on this kind of series, the readers may refer to four surveys [2, 3, 9, 16].

It is well known that \sqrt{p} are irrational numbers when p are arbitrary prime numbers. Several ones of this family of irrational numbers are closely related to π . For example, we have the following relations:

$$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}, \tag{2}$$

$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}, \tag{3}$$

$$\sin \frac{\pi}{10} = \frac{\sqrt{5}-1}{4}, \tag{4}$$

$$\frac{\sqrt{7}+1}{6\sqrt{3}}\pi = \sum_{k=0}^{\infty} \frac{(-1)^{\binom{k}{2}}}{2k+1} \left(\frac{3}{4+\sqrt{7}} \right)^k, \tag{5}$$

$$\frac{\pi}{3\sqrt{3}} = \sum_{k=0}^{\infty} \frac{1}{(649+180\sqrt{13})^k} \left\{ \frac{\sqrt{13}-3}{6k+1} - \frac{109\sqrt{13}-393}{6k+5} \right\}, \tag{6}$$

where (2)-(4) are proverbial and (5), (6) can be seen in Wei [15].

For a complex number x , define the shifted factorial to be

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1) \cdots (x+n-1) \quad \text{when} \quad n \in \mathbb{N}.$$

Following Bailey [1], define the hypergeometric series by

$${}_{1+r}F_s \left[\begin{matrix} a_0, & a_1, & \cdots, & a_r \\ & b_1, & \cdots, & b_s \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_r)_k}{(1)_k (b_1)_k \cdots (b_s)_k} z^k,$$

where $\{a_i\}_{i \geq 0}$ and $\{b_j\}_{j \geq 1}$ are complex parameters such that no zero factors appear in the denominators of the summand on the right hand side. Then three hypergeometric series identities can be stated as follows:

$${}_1F_0 \left[\begin{matrix} a \\ - \end{matrix} \middle| x \right] = \frac{1}{(1-x)^a} \quad \text{with } |x| < 1, \quad (7)$$

$${}_2F_1 \left[\begin{matrix} \frac{1}{2}a, \frac{1}{2} + \frac{1}{2}a \\ 1 + a \end{matrix} \middle| \frac{4x}{(1+x)^2} \right] = (1+x)^a \quad \text{with } \left| \frac{4x}{(1+x)^2} \right| < 1, \quad (8)$$

$${}_3F_2 \left[\begin{matrix} \frac{1}{3}a, \frac{1}{3} + \frac{1}{3}a, \frac{2}{3} + \frac{1}{3}a \\ \frac{1}{2} + \frac{1}{2}a, 1 + \frac{1}{2}a \end{matrix} \middle| \frac{27x}{4(1+x)^3} \right] = (1+x)^a \quad \text{with } \left| \frac{27x}{4(1+x)^3} \right| < 1, \quad (9)$$

where (7) is a well-known identity and (8), (9) can be found in Gessel and Stanton [6].

Inspired by the works just mentioned, we shall systematically explore series expansions for \sqrt{p} with p being prime numbers by means of Lemma 1 and (7)-(9). The structure of the paper is arranged as follows. We shall establish six theorems in Section 2. Then they and *Mathematica* program are utilized to produce concrete series expansions for \sqrt{p} in Sections 3-8.

2. SIX THEOREMS

Theorem 2. *Let p be a positive nonsquare integer and m, n be both positive integers satisfying $n^2 - pm^2 = 1$. Then*

$$\sqrt{p} = \frac{mp}{n} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(\frac{1}{pm^2 + 1} \right)^k.$$

Proof. The case $a = 1/2$ of (7) reads as

$$\frac{1}{\sqrt{1-x}} = \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} x^k \quad \text{with } |x| < 1. \quad (10)$$

Setting $x = 1/(pm^2 + 1)$ in (10), we achieve

$$\sqrt{\frac{pm^2 + 1}{pm^2}} = \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(\frac{1}{pm^2 + 1} \right)^k.$$

When $pm^2 + 1 = n^2$, the last equation becomes

$$\frac{n}{m} \frac{1}{\sqrt{p}} = \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(\frac{1}{pm^2 + 1} \right)^k.$$

Multiplying both sides by mp/n , we attain Theorem 2 to finish the proof. \square

Theorem 3. *Let p be a positive nonsquare integer and m, n be both positive integers provided that $n^2 - pm^2 = 1$. Then*

$$\sqrt{p} = \frac{n}{m} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(-\frac{1}{pm^2} \right)^k.$$

Proof. Taking $x = -1/pm^2$ in (10), we obtain

$$\sqrt{\frac{pm^2}{pm^2+1}} = \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(-\frac{1}{pm^2} \right)^k.$$

When $pm^2 + 1 = n^2$, the last equation creates

$$\frac{m}{n} \sqrt{p} = \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(-\frac{1}{pm^2} \right)^k.$$

Multiplying both sides by n/m , we get Theorem 3 to complete the proof. \square

Theorem 4. *Let p be a positive nonsquare integer and m, n be both positive integers satisfying $n^2 - pm^2 = 1$. Then*

$$\sqrt{p} = \frac{mp}{n} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left\{ \frac{4pm^2}{(pm^2+1)^2} \right\}^k.$$

Proof. The case $a = 1/2$ of (8) gives

$$\sqrt{1+x} = \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left\{ \frac{4x}{(1+x)^2} \right\}^k \quad \text{with} \quad \left| \frac{4x}{(1+x)^2} \right| < 1. \quad (11)$$

Fixing $x = 1/pm^2$ in (11), we gain

$$\sqrt{\frac{pm^2+1}{pm^2}} = \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left\{ \frac{4pm^2}{(pm^2+1)^2} \right\}^k.$$

When $pm^2 + 1 = n^2$, the last equation produces

$$\frac{n}{m} \frac{1}{\sqrt{p}} = \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left\{ \frac{4pm^2}{(pm^2+1)^2} \right\}^k.$$

Multiplying both sides by mp/n , we achieve Theorem 4 to finish the proof. \square

Theorem 5. *Let p be a positive nonsquare integer and m, n be both positive integers provided that $n^2 - pm^2 = 1$ and $p^2 m^4 > 4pm^2 + 4$. Then*

$$\sqrt{p} = \frac{n}{m} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left\{ -\frac{4(pm^2+1)}{p^2 m^4} \right\}^k.$$

Proof. Setting $x = -1/(pm^2+1)$ in (11), we attain

$$\sqrt{\frac{pm^2}{pm^2+1}} = \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left\{ -\frac{4(pm^2+1)}{p^2 m^4} \right\}^k.$$

When $pm^2 + 1 = n^2$, the last equation becomes

$$\frac{m}{n} \sqrt{p} = \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left\{ -\frac{4(pm^2+1)}{p^2 m^4} \right\}^k.$$

Multiplying both sides by n/m , we obtain Theorem 5 to complete the proof. \square

Theorem 6. *Let p be a positive nonsquare integer and m, n be both positive integers satisfying $n^2 - pm^2 = 1$ and $4(pm^2+1)^3 > 27p^2 m^4$. Then*

$$\sqrt{p} = \frac{mp}{n} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left\{ \frac{27p^2 m^4}{4(pm^2+1)^3} \right\}^k.$$

Proof. The case $a = 1/2$ of (9) offers

$$\sqrt{1+x} = \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left\{ \frac{27x}{4(1+x)^3} \right\}^k \quad \text{with} \quad \left| \frac{27x}{4(1+x)^3} \right| < 1. \quad (12)$$

Taking $x = 1/pm^2$ in (12), we get

$$\sqrt{\frac{pm^2+1}{pm^2}} = \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left\{ \frac{27p^2m^4}{4(pm^2+1)^3} \right\}^k.$$

When $pm^2+1 = n^2$, the last equation creates

$$\frac{n}{m} \frac{1}{\sqrt{p}} = \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left\{ \frac{27p^2m^4}{4(pm^2+1)^3} \right\}^k.$$

Multiplying both sides by mp/n , we gain Theorem 6 to finish the proof. \square

Theorem 7. *Let p be a positive nonsquare integer and m, n be both positive integers provided that $n^2 - pm^2 = 1$ and $4p^3m^6 > 27(pm^2+1)^2$. Then*

$$\sqrt{p} = \frac{n}{m} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left\{ -\frac{27(pm^2+1)^2}{4p^3m^6} \right\}^k.$$

Proof. Fixing $x = -1/(pm^2+1)$ in (12), we achieve

$$\sqrt{\frac{pm^2}{pm^2+1}} = \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left\{ -\frac{27(pm^2+1)^2}{4p^3m^6} \right\}^k.$$

When $pm^2+1 = n^2$, the last equation produces

$$\frac{m}{n} \sqrt{p} = \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left\{ -\frac{27(pm^2+1)^2}{4p^3m^6} \right\}^k.$$

Multiplying both sides by n/m , we attain Theorem 7 to complete the proof. \square

3. SERIES EXPANSIONS FOR $\sqrt{2}$

Setting $p = 2$ in (1), we obtain

$$x^2 - 2y^2 = 1. \quad (13)$$

It is easy to know that $x_1 = 3, y_1 = 2$ is the solution to (13). So

$$\begin{cases} x_s = \sum_{k=0}^{\lfloor \frac{s}{2} \rfloor} \binom{s}{2k} 2^{3k} 3^{s-2k}, \\ y_s = \sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} \binom{s}{1+2k} 2^{1+3k} 3^{s-1-2k} \end{cases}$$

is also the solution of (13) thanks to Lemma 1. Now we choose $x_4 = 577, y_4 = 408$ and $x_7 = 114243, y_7 = 80782$ to give 12 series expansions for $\sqrt{2}$.

Substituting $p=2$, $n = x_4 = 577$ and $m = y_4 = 408$ into Theorems 2-7, we get the following six series expansions for $\sqrt{2}$:

$$\begin{aligned}\sqrt{2} &= \frac{816}{577} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(\frac{1}{332929} \right)^k, \\ \sqrt{2} &= \frac{577}{408} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(-\frac{1}{332928} \right)^k, \\ \sqrt{2} &= \frac{816}{577} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(\frac{1331712}{110841719041} \right)^k, \\ \sqrt{2} &= \frac{577}{408} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(-\frac{332929}{27710263296} \right)^k, \\ \sqrt{2} &= \frac{816}{577} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(\frac{748177108992}{36902422678601089} \right)^k, \\ \sqrt{2} &= \frac{577}{408} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(-\frac{110841719041}{5466976319176704} \right)^k.\end{aligned}$$

Substituting $p=2$, $n = x_7 = 114243$ and $m = y_7 = 80782$ into Theorems 2-7, we gain the following six series expansions for $\sqrt{2}$:

$$\begin{aligned}\sqrt{2} &= \frac{161564}{114243} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(\frac{1}{13051463049} \right)^k, \\ \sqrt{2} &= \frac{114243}{80782} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(-\frac{1}{13051463048} \right)^k, \\ \sqrt{2} &= \frac{161564}{114243} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(\frac{52205852192}{170340687719412376401} \right)^k, \\ \sqrt{2} &= \frac{114243}{80782} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(-\frac{13051463049}{42585171923327362576} \right)^k, \\ \sqrt{2} &= \frac{161564}{114243} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(\frac{42585171923327362576}{82340562648561433725590040987} \right)^k, \\ \sqrt{2} &= \frac{114243}{80782} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(-\frac{4599198568424134162827}{8892780764000546589887393466368} \right)^k.\end{aligned}$$

Numerous different series expansions for $\sqrt{2}$ can be derived in the same way. Due to limit of space, the corresponding results will not be displayed in the paper. The discuss is also adapt to series expansions for \sqrt{p} with $p > 2$.

4. SERIES EXPANSIONS FOR $\sqrt{3}$

Taking $p = 3$ in (1), we have

$$x^2 - 3y^2 = 1. \quad (14)$$

It is not difficult to see that $x_1 = 2, y_1 = 1$ is the solution to (14). Thus

$$\begin{cases} x_s = \sum_{k=0}^{\lfloor \frac{s}{2} \rfloor} \binom{s}{2k} 3^k 2^{s-2k}, \\ y_s = \sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} \binom{s}{1+2k} 3^k 2^{s-1-2k} \end{cases}$$

is also the solution of (14) according to Lemma 1. Now we select $x_5 = 362, y_5 = 209$ and $x_9 = 70226, y_9 = 40545$ to create 12 series expansions for $\sqrt{3}$.

Substituting $p=3, n = x_5 = 362$ and $m = y_5 = 209$ into Theorems 2-7, we achieve the following six series expansions for $\sqrt{3}$:

$$\begin{aligned}\sqrt{3} &= \frac{627}{362} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(\frac{1}{131044} \right)^k, \\ \sqrt{3} &= \frac{362}{209} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(-\frac{1}{131043} \right)^k, \\ \sqrt{3} &= \frac{627}{362} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(\frac{131043}{4293132484} \right)^k, \\ \sqrt{3} &= \frac{362}{209} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(-\frac{524176}{17172267849} \right)^k, \\ \sqrt{3} &= \frac{627}{362} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(\frac{463651231923}{9001428051732736} \right)^k, \\ \sqrt{3} &= \frac{362}{209} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(-\frac{4293132484}{83344647990241} \right)^k.\end{aligned}$$

Substituting $p=3, n = x_9 = 70226$ and $m = y_9 = 40545$ into Theorems 2-7, we attain the following six series expansions for $\sqrt{3}$:

$$\begin{aligned}\sqrt{3} &= \frac{121635}{70226} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(\frac{1}{4931691076} \right)^k, \\ \sqrt{3} &= \frac{70226}{40545} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(-\frac{1}{4931691075} \right)^k, \\ \sqrt{3} &= \frac{121635}{70226} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(\frac{4931691075}{6080394217274509444} \right)^k, \\ \sqrt{3} &= \frac{70226}{40545} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(-\frac{19726764304}{24321576859234655625} \right)^k, \\ \sqrt{3} &= \frac{121635}{70226} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(\frac{656682575199335701875}{479786014398315252276040347904} \right)^k, \\ \sqrt{3} &= \frac{70226}{40545} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(-\frac{6080394217274509444}{4442463093578299350981890625} \right)^k.\end{aligned}$$

5. SERIES EXPANSIONS FOR $\sqrt{5}$

Fixing $p = 5$ in (1), we obtain

$$x^2 - 5y^2 = 1. \quad (15)$$

It is ordinary to find that $x_1 = 9, y_1 = 4$ is the solution to (15). Therefore

$$\begin{cases} x_s = \sum_{k=0}^{\lfloor \frac{s}{2} \rfloor} \binom{s}{2k} 5^k 4^{2k} 9^{s-2k}, \\ y_s = \sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} \binom{s}{1+2k} 5^k 4^{1+2k} 9^{s-1-2k} \end{cases}$$

is also the solution of (15) in accordance with Lemma 1. Now we choose $x_3 = 2889$, $y_3 = 1292$ and $x_4 = 51841$, $y_4 = 23184$ to produce 12 series expansions for $\sqrt{5}$.

Substituting $p=5$, $n = x_3 = 2889$ and $m = y_3 = 1292$ into Theorems 2-7, we get the following six series expansions for $\sqrt{5}$:

$$\begin{aligned}\sqrt{5} &= \frac{6460}{2889} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(\frac{1}{8346321} \right)^k, \\ \sqrt{5} &= \frac{2889}{1292} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(-\frac{1}{8346320} \right)^k, \\ \sqrt{5} &= \frac{6460}{2889} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(\frac{33385280}{69661074235041} \right)^k, \\ \sqrt{5} &= \frac{2889}{1292} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(-\frac{8346321}{17415264385600} \right)^k, \\ \sqrt{5} &= \frac{6460}{2889} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(\frac{17415264385600}{21533840250758579043} \right)^k, \\ \sqrt{5} &= \frac{2889}{1292} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(-\frac{1880849004346107}{2325653911149135872000} \right)^k.\end{aligned}$$

Substituting $p=5$, $n = x_4 = 51841$ and $m = y_4 = 23184$ into Theorems 2-7, we gain the following six series expansions for $\sqrt{5}$:

$$\begin{aligned}\sqrt{5} &= \frac{115920}{51841} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(\frac{1}{2687489281} \right)^k, \\ \sqrt{5} &= \frac{51841}{23184} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(-\frac{1}{2687489280} \right)^k, \\ \sqrt{5} &= \frac{115920}{51841} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(\frac{10749957120}{7222598635489896961} \right)^k, \\ \sqrt{5} &= \frac{51841}{23184} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(-\frac{2687489281}{1805649657528729600} \right)^k, \\ \sqrt{5} &= \frac{115920}{51841} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(\frac{48752540753275699200}{19410656413844324266481975041} \right)^k, \\ \sqrt{5} &= \frac{51841}{23184} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(-\frac{7222598635489896961}{2875652798840967165640704000} \right)^k.\end{aligned}$$

6. SERIES EXPANSIONS FOR $\sqrt{7}$

Setting $p = 7$ in (1), we have

$$x^2 - 7y^2 = 1. \quad (16)$$

It is easy to know that $x_1 = 8$, $y_1 = 3$ is the solution to (16). So

$$\begin{cases} x_s = \sum_{k=0}^{\lfloor \frac{s}{2} \rfloor} \binom{s}{2k} 7^k 3^{2k} 8^{s-2k}, \\ y_s = \sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} \binom{s}{1+2k} 7^k 3^{1+2k} 8^{s-1-2k} \end{cases}$$

is also the solution of (16) thanks to Lemma 1. Now we select $x_3 = 2024, y_3 = 765$ and $x_4 = 32257, y_4 = 12192$ to give 12 series expansions for $\sqrt{7}$.

Substituting $p=7, n = x_3 = 2024$ and $m = y_3 = 765$ into Theorems 2-7, we achieve the following six series expansions for $\sqrt{7}$:

$$\begin{aligned}\sqrt{7} &= \frac{5355}{2024} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(\frac{1}{4096576} \right)^k, \\ \sqrt{7} &= \frac{2024}{765} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(-\frac{1}{4096575} \right)^k, \\ \sqrt{7} &= \frac{5355}{2024} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(\frac{4096575}{4195483730944} \right)^k, \\ \sqrt{7} &= \frac{2024}{765} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(-\frac{16386304}{16781926730625} \right)^k, \\ \sqrt{7} &= \frac{5355}{2024} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(\frac{453112021726875}{274993887369210363904} \right)^k, \\ \sqrt{7} &= \frac{2024}{765} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(-\frac{4195483730944}{2546237833204078125} \right)^k.\end{aligned}$$

Substituting $p=7, n = x_4 = 32257$ and $m = y_4 = 12192$ into Theorems 2-7, we attain the following six series expansions for $\sqrt{7}$:

$$\begin{aligned}\sqrt{7} &= \frac{85344}{32257} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(\frac{1}{1040514049} \right)^k, \\ \sqrt{7} &= \frac{32257}{12192} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(-\frac{1}{1040514048} \right)^k, \\ \sqrt{7} &= \frac{85344}{32257} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(\frac{4162056192}{1082669486166374401} \right)^k, \\ \sqrt{7} &= \frac{32257}{12192} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(-\frac{1040514049}{270667371021336576} \right)^k, \\ \sqrt{7} &= \frac{85344}{32257} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(\frac{7308019017576087552}{1126532810779723715634459649} \right)^k, \\ \sqrt{7} &= \frac{32257}{12192} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(-\frac{1082669486166374401}{166893749263957816334352384} \right)^k.\end{aligned}$$

7. SERIES EXPANSIONS FOR $\sqrt{11}$

Taking $p = 11$ in (1), we obtain

$$x^2 - 11y^2 = 1. \quad (17)$$

It is not difficult to see that $x_1 = 10, y_1 = 3$ is the solution to (17). Thus

$$\begin{cases} x_s = \sum_{k=0}^{\lfloor \frac{s}{2} \rfloor} \binom{s}{2k} 11^k 3^{2k} 10^{s-2k}, \\ y_s = \sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} \binom{s}{1+2k} 11^k 3^{1+2k} 10^{s-1-2k} \end{cases}$$

is also the solution of (17) according to Lemma 1. Now we choose $x_3 = 3970, y_3 = 1197$ and $x_4 = 79201, y_4 = 23880$ to create 12 series expansions for $\sqrt{11}$.

Substituting $p=11, n = x_3 = 3970$ and $m = y_3 = 1197$ into Theorems 2-7, we get the following six series expansions for $\sqrt{11}$:

$$\begin{aligned}\sqrt{11} &= \frac{13167}{3970} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(\frac{1}{15760900} \right)^k, \\ \sqrt{11} &= \frac{3970}{1197} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(-\frac{1}{15760899} \right)^k, \\ \sqrt{11} &= \frac{13167}{3970} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(\frac{15760899}{62101492202500} \right)^k, \\ \sqrt{11} &= \frac{3970}{1197} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(-\frac{63043600}{248405937288201} \right)^k, \\ \sqrt{11} &= \frac{13167}{3970} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(\frac{6706960306781427}{15660406535270116000000} \right)^k, \\ \sqrt{11} &= \frac{3970}{1197} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(-\frac{62101492202500}{145003736614802587137} \right)^k.\end{aligned}$$

Substituting $p=11, n = x_4 = 79201$ and $m = y_4 = 23880$ into Theorems 2-7, we gain the following six series expansions for $\sqrt{11}$:

$$\begin{aligned}\sqrt{11} &= \frac{262680}{79201} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(\frac{1}{6272798401} \right)^k, \\ \sqrt{11} &= \frac{79201}{23880} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(-\frac{1}{6272798400} \right)^k, \\ \sqrt{11} &= \frac{262680}{79201} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(\frac{25091193600}{39347999779588156801} \right)^k, \\ \sqrt{11} &= \frac{79201}{23880} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(-\frac{6272798401}{9836999941760640000} \right)^k, \\ \sqrt{11} &= \frac{262680}{79201} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(\frac{265598998427537280000}{246822070099948942419850075201} \right)^k, \\ \sqrt{11} &= \frac{79201}{23880} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(-\frac{39347999779588156801}{36566232589911843422208000000} \right)^k.\end{aligned}$$

8. SERIES EXPANSIONS FOR $\sqrt{13}$

Fixing $p = 13$ in (1), we have

$$x^2 - 13y^2 = 1. \tag{18}$$

We can find that $x_1 = 649, y_1 = 180$ is the solution to (18) by means of the commands of *Mathematica*:

```

M = 180;
aa[m_] :=  $\sqrt{13m^2 + 1}$ ;
For[i = 1, i <= M,
  x = aa[i];
  If[Mod[x, 1] == 0, Print[ $\sqrt{13i^2 + 1}$ ]]
  i++
]
649.

```

Therefore

$$\begin{cases} x_s = \sum_{k=0}^{\lfloor \frac{s}{2} \rfloor} \binom{s}{2k} 13^k 180^{2k} 649^{s-2k}, \\ y_s = \sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} \binom{s}{1+2k} 13^k 180^{1+2k} 649^{s-1-2k} \end{cases}$$

is also the solution of (18) in accordance with Lemma 1. Now we select $x_1 = 649, y_1 = 180$ and $x_2 = 842401, y_2 = 233640$ to produce 12 series expansions for $\sqrt{13}$.

Substituting $p=13, n = x_1 = 649$ and $m = y_1 = 180$ into Theorems 2-7, we achieve the following six series expansions for $\sqrt{13}$:

$$\begin{aligned} \sqrt{13} &= \frac{2340}{649} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(\frac{1}{421201} \right)^k, \\ \sqrt{13} &= \frac{649}{180} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(-\frac{1}{421200} \right)^k, \\ \sqrt{13} &= \frac{2340}{649} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(\frac{1684800}{177410282401} \right)^k, \\ \sqrt{13} &= \frac{649}{180} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(-\frac{421201}{44352360000} \right)^k, \\ \sqrt{13} &= \frac{2340}{649} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(\frac{1197513720000}{74725388357583601} \right)^k, \\ \sqrt{13} &= \frac{649}{180} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \left(-\frac{177410282401}{11070349056000000} \right)^k. \end{aligned}$$

Substituting $p=13, n = x_2 = 842401$ and $m = y_2 = 233640$ into Theorems 2-7, we attain the following six series expansions for $\sqrt{13}$:

$$\begin{aligned} \sqrt{13} &= \frac{3037320}{842401} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(\frac{1}{709639444801} \right)^k, \\ \sqrt{13} &= \frac{842401}{233640} \sum_{k=0}^{\infty} \frac{(1/2)_k}{k!} \left(-\frac{1}{709639444800} \right)^k, \\ \sqrt{13} &= \frac{3037320}{842401} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(\frac{2838557779200}{503588141617471525929601} \right)^k, \end{aligned}$$

$$\begin{aligned}
\sqrt{13} &= \frac{842401}{233640} \sum_{k=0}^{\infty} \frac{(1/4)_k (3/4)_k}{k! (3/2)_k} \left(-\frac{709639444801}{125897035404013061760000} \right)^k, \\
\sqrt{13} &= \frac{3037320}{842401} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \\
&\quad \times \left(\frac{3399219955908352667520000}{357366009225789855782108329051454401} \right)^k, \\
\sqrt{13} &= \frac{842401}{233640} \sum_{k=0}^{\infty} \frac{(1/2)_k (1/6)_k (5/6)_k}{k! (3/4)_k (5/4)_k} \\
&\quad \times \left(-\frac{503588141617471525929601}{52943112477670976497371561984000000} \right)^k.
\end{aligned}$$

In this paper, we establish numerous series expansions for \sqrt{p} with $p = 2, 3, 5, 7, 11, 13$. When p are other prime numbers, series expansions for \sqrt{p} , which converge fast, can also be established in the same way.

Acknowledgments

The work is supported by the Natural Science Foundations of China (Nos. 11301120, 11201241).

REFERENCES

- [1] W.N. Bailey, Generalized Hypergeometric Series, Cambridge University Press, Cambridge, 1935.
- [2] D.H. Bailey, J.M. Borwein, Experimental mathematics: examples, methods and implications, Notices Amer. Math. Soc. 52 (2005), 502-514.
- [3] N.D. Baruah, B.C. Berndt, H.H. Chan, Ramanujan's series for $1/\pi$: a survey, Amer. Math. Monthly 116 (2009), 567-587.
- [4] W. Chu, π -formulas implied by Dougall's summation theorem for ${}_5F_4$ -series, Ramanujan J. 26 (2011), 251-255.
- [5] W. Chu, Accelerating Dougall's ${}_5F_4$ -series and infinite series involving π , Math. Comput. 83 (2014), 474-512.
- [6] I. Gessel, D. Stanton, Strange evaluations of hypergeometric series, SIAM. Math. Anal. 13 (1982), 295-308.
- [7] J. Guillera, About a new kind of Ramanujan-type series, Exper. Math. 12 (2003), 507-510.
- [8] J. Guillera, Generators of some Ramanujan formulas, Ramanujan J. 11 (2006), 41-48.
- [9] J. Guillera, History of the formulas and algorithms for π (Spanish), Gac. R. Soc. Mat. Esp. 10 (2007), 159-178.
- [10] J. Guillera, Hypergeometric identities for 10 extended Ramanujan-type series, Ramanujan J. 15 (2008), 219-234.
- [11] J. Guillera, A new Ramanujan-like series for $1/\pi^2$, Ramanujan J. 26 (2011), 369-374.
- [12] Z. Liu, Gauss summation and Ramanujan type series for $1/\pi$, Int. J. Number Theory 8 (2012), 289-297.
- [13] Z. Liu, A summation formula and Ramanujan type series, J. Math. Anal. Appl. 389 (2012), 1059-1065.
- [14] C. Wei, D. Gong, Extensions of Ramanujan's two formulas for $1/\pi$, J. Number Theory 133 (2013), 2206-2216.

- [15] C. Wei, Several BBP-type formulas for π , *Integral Transforms Spec. Funct.* 26 (2015), 315-324.
- [16] W. Zudilin, Ramanujan-type formulas for $1/\pi$: A second wind?, in *Modular Forms and String Duality*, N. Yui, H. Verrill, C.F. Doran, eds., Fields Institute Communications, vol. 54, American Mathematical Society & The fields Institute for Research in Mathematical Sciences, Providence, RI, 2008, pp. 179-188.